Reductions and hodograph solutions of the dispersionless KP hierarchy

2002 J. Phys. A: Math. Gen. 35 401

(http://iopscience.iop.org/0305-4470/35/2/316)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 401–417 PII: S0305-4470(02)28836-8

Reductions and hodograph solutions of the dispersionless KP hierarchy*

Manuel Mañas[1], **Luis Martínez Alonso**[1] **and Elena Medina**[2]

[1] Departamento de Física Teórica II, Universidad Complutense, E28040 Madrid, Spain
[2] Departamento de Matemáticas, Universidad de Cádiz, E11510 Puerto Real, Cádiz, Spain

E-mail: manuel@darboux.fis.ucm.es, luism@eucmos.sim.ucm.es and elena.medina@uca.es

Received 17 September 2001
Published 4 January 2002
Online at stacks.iop.org/JPhysA/35/401

Abstract

A general scheme for analysing reductions of dispersionless integrable hierarchies is presented. It is based on a method for determining the S-function by means of a system of first-order differential equations. Compatibility systems of nonlinear partial differential equations of Bourlet type characterizing both reductions and hodograph solutions of the dKP hierarchy are obtained. Wide classes of illustrative explicit examples are exhibited.

PACS numbers: 02.30.−f, 02.30.Jr, 02.40.−k
Mathematics Subject Classification: 58B20

1. Introduction

One of the most important problems in the theory of integrable hierarchies of nonlinear evolution equations is the analysis of their reductions. Over the last decade this subject has registered particularly increasing activity in connection with the hierarchies of dispersionless integrable systems. These systems have important applications to several fields such as, for instance, the dispersionless limit of solutions of integrable models on the zero-phase domains [1, 2], the classification problem of topological field theory [3–5], the study of systems of hydrodynamic type [6] or the theory of conformal maps [7–9]. Several strategies have been proposed to deal with the solutions of dispersionless hierarchies. The use of reductions in this context is a relevant step within the hodograph method of solution [6, 10, 11], which can be conveniently illustrated when applied to the dispersionless KP (dKP) hierarchy [10–13]

$$\frac{\partial z}{\partial t_n} = \{\Omega_n, z\} \qquad \Omega_n := (z^n)_+ \qquad n \geqslant 1. \tag{1}$$

Here $z = z(p, \boldsymbol{t})$ is a function depending on a complex variable p and an infinite set of complex time parameters $\boldsymbol{t} := (x := t_1, t_2, \ldots)$, that admits an expansion

$$z = p + \sum_{n=1}^{\infty} \frac{a_n(\boldsymbol{t})}{p^n} \qquad p \to \infty \tag{2}$$

* Partially supported by CICYT proyecto PB98–0821.

$\{\cdot,\cdot\}$ is the Poisson bracket

$$\{F_1, F_2\} := \frac{\partial F_1}{\partial p}\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial p}$$

and $\Omega_n = (z^n)_+$ denotes the polynomial part of z^n as a function of p

$$(z)_+ = p \qquad (z^2)_+ = p^2 + 2a_1 \qquad (z^3)_+ = p^3 + 3p\,a_1 + 3a_2$$

$$(z^4)_+ = p^4 + 4p^2 a_1 + 4p\,a_2 + 6a_1^2 + 4a_3.$$

For $n = 2$, (1) leads to the Benney moment equations [13, 14]

$$\frac{\partial a_{n+1}}{\partial t} + \frac{\partial a_{n+2}}{\partial x} + na_n\frac{\partial a_1}{\partial x} = 0 \qquad t := -2t_2 \tag{3}$$

and the compatibility equations for (1)

$$\frac{\partial \Omega_m}{\partial t_n} - \frac{\partial \Omega_n}{\partial t_m} + \{\Omega_m, \Omega_n\} = 0 \qquad m \neq n \tag{4}$$

form a hierarchy of nonlinear partial differential equations. For instance, by setting $m = 3, n = 2$ we get the dKP equation (Zabolotskaya–Khokhlov equation)

$$(u_t + 3uu_x)_x = \tfrac{3}{4}u_{yy} \qquad u := -a_1 \quad t := t_3 \quad y := t_2 \tag{5}$$

and for $m = 4,\ n = 2$ one gets

$$v_x = \tfrac{1}{2}u_y \qquad \left(\tfrac{1}{2}v_y + uu_x\right)_y = \left(\tfrac{1}{2}u_t + 3uu_y + 2vu_x\right)_x \tag{6}$$

with $v := -a_2, t := t_4$ and u and y are as in (5).

There are several well-known examples of explicit reductions of the dKP hierarchy in which $z = z(p, \boldsymbol{t})$ depends on $\boldsymbol{t}$ through only finitely many functions [3]. A scheme to deal with general reductions, without requiring knowledge of the explicit form of $z = z(p, \boldsymbol{t})$, is given by Kodama and Gibbons in [10, 11, 6]. They define an N-reduction of the dKP hierarchy as a function $z = z(p, \boldsymbol{u})$ of the form (2), depending on $\boldsymbol{t}$ through N functions $\boldsymbol{u} = (u_1(\boldsymbol{t}), \ldots, u_N(\boldsymbol{t}))$ satisfying a compatible system of hydrodynamic-type (HT) equations

$$\frac{\partial \boldsymbol{u}}{\partial t_n} = A_n(\boldsymbol{u})\frac{\partial \boldsymbol{u}}{\partial x} \qquad n > 1 \tag{7}$$

such that $z = z(p, \boldsymbol{u}(\boldsymbol{t}))$ solves (1). Here A_n are $N \times N$ matrix functions depending on $\boldsymbol{u}$ only. Furthermore, if A_2 has N different eigenvalues and $u_{i,x}$, $i = 1, \ldots, N$, are independent, the matrices A_n are necessarily given by the functions $\frac{\partial \Omega_n}{\partial p}$ evaluated at $p = A_2/2$. The corresponding HT equations (7) turn out to be diagonalized by means of a set of Riemann invariants provided by the turning points $z_i := z(p_i(\boldsymbol{u}), \boldsymbol{u})$ of the function $z(p, \boldsymbol{u})$.

In [8, 9] Gibbons and Tsarev consider the N-reductions of the Benney moment equations (3). They take the N first moments of $z = z(p, \boldsymbol{u})$ as the functions $\boldsymbol{u}$ ($u_i := a_i,\ i = 1, \ldots, N$), while the higher moments are assumed to be functions $a_n = a_n(\boldsymbol{u})$, $n > N$, of them. As a consequence (3) becomes an HT system for $\boldsymbol{u}$ (involving the function $a_{N+1}(\boldsymbol{u})$) and a over-determined system for the functions $a_n(\boldsymbol{u}), n > N$. The compatibility conditions of the latter reduce to a system of $N(N-1)/2$ second-order differential equations for $a_{N+1}(\boldsymbol{u})$, the solutions of which determine diagonalizable HT systems for $\boldsymbol{u}$. Note that these HT systems play the role of the $n = 2$ flows in (7) with a diagonalizable matrix A_2. In this sense the results of [8, 9] complement those of [6, 10, 11], so that the Gibbons–Tsarev analysis applies to the general reduction problem of the dKP hierarchy.

The starting point of this work is the characterization of the reductions of the dKP hierarchy in terms of systems of differential equations for $p = p(z, \boldsymbol{u})$ of the form

$$\frac{\partial p}{\partial u_i} = \sum_{j=1}^{N} \frac{r_{ij}(\boldsymbol{u})}{p - p_j(\boldsymbol{u})} \qquad i = 1, \ldots, N \tag{8}$$

satisfying the following compatibility conditions:

$$\begin{aligned}
r_{ik}\frac{\partial p_k}{\partial u_j} - r_{jk}\frac{\partial p_k}{\partial u_i} &= \sum_{l\neq k}\frac{r_{jl}r_{ik} - r_{il}r_{jk}}{p_k - p_l}\\
\frac{\partial r_{ik}}{\partial u_j} - \frac{\partial r_{jk}}{\partial u_i} &= 2\sum_{l\neq k}\frac{r_{jk}r_{il} - r_{ik}r_{jl}}{(p_k - p_l)^2}.
\end{aligned} \tag{9}$$

This class includes, in particular, the standard reductions associated with functional constraints for $z = z(p, \boldsymbol{u})$ such as

(1) Gel'fand–Dikii reductions

$$z^{N+1} = p^{N+1} + u_1 p^{N-1} + \cdots + u_N.$$

(2) Zakharov reductions

$$z = p + \sum_{i=1}^{M}\frac{h_i}{p - v_i}.$$

(3) Kodama reductions

$$z^{N+1} = p^{N+1} + u_1 p^{N-1} + \cdots + u_N + \frac{v_1}{p - v_0} + \cdots + \frac{v_M}{(p - v_0)^M}.$$

The basic ingredient of our analysis is a method for characterizing the S-function for the reductions (8) of the dKP hierarchy in terms of a system of differential equations. The corresponding compatibility conditions together with (9) constitute a system of first-order nonlinear differential equations of Bourlet type. It characterizes both the reductions and the hodograph solutions of the dKP hierarchy.

2. Reductions of the dKP hierarchy

2.1. The S-function

From (4) it follows [13] that there exists a function $S = S(z, \boldsymbol{t})$, such that

$$\frac{\partial S(z)}{\partial t_n} = \Omega_n(p, \boldsymbol{t}) \qquad n \geqslant 1. \tag{10}$$

This function is a basic object of the dKP theory and it will henceforth be referred to as *the S-function*. Without loss of generality it can be assumed that S has an expansion

$$S(z, \boldsymbol{t}) = \sum_{n\geqslant 1} z^n t_n + \sum_{n\geqslant 1}\frac{S_n(\boldsymbol{t})}{z^n} \qquad z \to \infty. \tag{11}$$

If S satisfies (10) and (11), then by setting $n = 1$ in (10) one finds p as a function $p = p(z, \boldsymbol{t})$ of the form

$$p = z + \sum_{n\geqslant 1}\frac{b_n(\boldsymbol{t})}{z^n} \qquad b_n := \frac{\partial S_n}{\partial x} \tag{12}$$

and it can be proved [13] that the inverted series determines a solution $z = z(p, \boldsymbol{t})$ of the dKP hierarchy. The conditions (10) which characterize an S-function constitute a system of compatible Hamilton–Jacobi-type equations

$$\frac{\partial S}{\partial t_n} = \Omega_n\left(\frac{\partial S}{\partial x}, \boldsymbol{t}\right) \qquad n \geqslant 2$$

which represents the *semiclassical limit* of the linear system for the wavefunction of the standard KP hierarchy.

From (11) and (12) it is clear that a function S with an expansion of the form (11) satisfies (10) if and only if the derivatives $\frac{\partial S(z)}{\partial t_n}$, considered as power series of p, have no terms with negative powers of p. In other words, the conditions (10) are equivalent to

$$\left(\frac{\partial S(z)}{\partial t_n}\right)_- = 0 \qquad n \geqslant 1. \tag{13}$$

Henceforth we will use S as a function of either z or p and will denote by $S(z)$ or $S(p)$ the corresponding functions $(S(z, \boldsymbol{t}) = S(p(z, \boldsymbol{t}), \boldsymbol{t}))$. Furthermore, we will denote by $S(p) = S_+(p) + S_-(p)$ the decomposition of $S(p)$ in terms of positive and negative powers of p. Obviously, from (11) and (12) we deduce

$$S_+(p) = \sum_{n\geqslant 1} \Omega_n t_n. \tag{14}$$

Hence the conditions (13) for S can be rewritten in the following form

$$\left(\frac{\partial S(p)}{\partial p}\frac{\partial p}{\partial t_n} + \frac{\partial S_-(p)}{\partial t_n}\right)_- = 0 \qquad n \geqslant 1 \tag{15}$$

which will be useful in what follows.

2.2. *N-reductions*

We will consider N-reductions of the dKP hierarchy determined by systems of equations for $p = p(z, \boldsymbol{u})$ of the form

$$\frac{\partial p}{\partial u_i} = R_i(p, \boldsymbol{u}) \qquad i = 1, \ldots, N \tag{16}$$

or, equivalently, in terms of $z = z(p, \boldsymbol{u})$

$$\frac{\partial z}{\partial u_i} + R_i(p, \boldsymbol{u})\frac{\partial z}{\partial p} = 0 \qquad i = 1, \ldots, N. \tag{17}$$

The following conditions for the functions R_i will be assumed:

(i) The functions R_i are rational functions of p which have singularities only at N simple poles $p_i = p_i(\boldsymbol{u})$, $i = 1, \ldots, N$, and vanish at $p = \infty$. Therefore they can be expanded as

$$R_i(p, \boldsymbol{u}) = \sum_{j=1}^{N} \frac{r_{ij}(\boldsymbol{u})}{p - p_j(\boldsymbol{u})}. \tag{18}$$

(ii) The functions R_i satisfy the compatibility conditions for (17)

$$\frac{\partial R_i}{\partial u_j} - \frac{\partial R_j}{\partial u_i} + R_j\frac{\partial R_i}{\partial p} - R_i\frac{\partial R_j}{\partial p} = 0 \qquad i \neq j. \tag{19}$$

We are going to prove that under these assumptions the solutions $z = z(p, \boldsymbol{u})$ of (17) define N-reductions of the dKP hierarchy. Our method consists in deriving hodograph relations which determine a class of functions $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{t})$ for which an S-function for $z = z(p, \boldsymbol{u}(\boldsymbol{t}))$ exists.

To this end let us consider the conditions (15) for S and assume that not only $p(z, \boldsymbol{t})$ but also $S_-(p)$ depends on $\boldsymbol{t}$ through the functions $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{t})$. In this way, (15) holds if

$$\left(\frac{\partial S(p)}{\partial p}\frac{\partial p}{\partial u_i} + \frac{\partial S_-(p)}{\partial u_i}\right)_- = 0$$

or, equivalently, from the reduction condition (16)

$$\left(\frac{\partial S(p)}{\partial p}R_i + \frac{\partial S_-(p)}{\partial u_i}\right)_- = 0. \tag{20}$$

We will look for a S-function such that

$$\frac{\partial S}{\partial p}(p_i) = 0. \tag{21}$$

Let us denote by $E = E(p, \boldsymbol{u})$ any entire function in p satisfying

$$E(p_i, \boldsymbol{u}) = F_i(\boldsymbol{u}) \qquad i = 1, \dots, N$$

where

$$F_i(\boldsymbol{u}) := \frac{\partial S_-}{\partial p}(p_i). \tag{22}$$

Then by decomposing

$$\frac{\partial S}{\partial p}R_i + \frac{\partial S_-}{\partial u_i} = \left(\frac{\partial S_+}{\partial p} + E\right)R_i + \left(\frac{\partial S_-}{\partial p} - E\right)R_i + \frac{\partial S_-}{\partial u_i}$$

and by taking into account that according to our hypothesis

$$\left(\left(\frac{\partial S_+}{\partial p} + E\right)R_i\right)_- = 0$$

we conclude that (20) is equivalent to the following system of differential equations for S_-

$$\frac{\partial S_-(p)}{\partial u_i} + R_i\frac{\partial S_-(p)}{\partial p} = (ER_i)_-. \tag{23}$$

We note that they imply

$$\operatorname{Res}\left(R_i\frac{\partial S_-}{\partial p}, p_j\right) = \operatorname{Res}((ER_i)_-, p_j) = \operatorname{Res}(ER_i, p_j)$$

so that (22) is satisfied by the solutions of (23). Moreover, by using (19) one finds that the compatibility conditions for (23) are

$$\frac{\partial (ER_i)_-}{\partial u_j} - \frac{\partial (ER_j)_-}{\partial u_i} + R_j\frac{\partial (ER_i)_-}{\partial p} - R_i\frac{\partial (ER_j)_-}{\partial p} = 0 \qquad i \neq j. \tag{24}$$

By taking into account that

$$(ER_j)_- = \sum_{k=1}^{N}\frac{r_{jk}F_k}{p - p_k}$$

one sees that (24) represent a set of consistency conditions for the functions F_j.

To sum up, if the functions $R_i(p, \boldsymbol{u})$ and $F_i(\boldsymbol{u})$ ($i = 1, \dots N$) satisfy (19) and (24), a solution $z = z(p, \boldsymbol{u})$ of (17) of the form

$$z = p + \sum_{n\geqslant 1}\frac{a_n(\boldsymbol{u})}{p^n} \tag{25}$$

determines an N-reduction of the dKP hierarchy. Indeed, from (14) and (25) we determine $S_+(p)$ in terms of the coefficients $a_n(\boldsymbol{u})$ and then, by using the conditions (21) as N implicit equations

$$\frac{\partial S_+}{\partial p}(p_i) + F_i(\boldsymbol{u}) = 0$$

or, equivalently

$$\sum_{n=1}^{\infty} v_{in}(\boldsymbol{u})t_n + F_i(\boldsymbol{u}) = 0 \qquad v_{in} := \frac{\partial \Omega_n}{\partial p}(p_i) \tag{26}$$

we characterize a class of functions $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{t})$ for which $z = z(p, \boldsymbol{u}(\boldsymbol{t}))$ admits an S-function. Observe that the series

$$S_-(p) = \sum_{n\geqslant 1} \frac{S_n(\boldsymbol{u})}{p^n}$$

can be recursively determined from (23). Consequently, $z = z(p, \boldsymbol{u}(\boldsymbol{t}))$ solves the equations (1) of the dKP hierarchy. In view of the form of the implicit relations (26) these solutions will henceforth be called *hodograph solutions*.

Obviously, the choice $F_i \equiv 0,\ i = 1, \ldots, N$ corresponds to $S_- \equiv 0$ of (23). On the other hand, if $(R_i, F_i), i = 1, \ldots, N$ is a solution of the compatibility conditions (19) and (24) and $z = z(p, \boldsymbol{u})$ is the associated solution of (17), then, for every entire function $P = P(z)$

$$\tilde{F}_i := F_i + \left.\frac{\partial P(z)_+}{\partial p}\right|_{p=p_i} \tag{27}$$

is a new solution of (24). The proof of this property follows from the fact that (17) implies

$$\frac{\partial P(z)}{\partial u_i} + R_i(p, \boldsymbol{u})\frac{\partial P(z)}{\partial p} = 0$$

so that

$$\frac{\partial P(z)_-}{\partial u_i} + R_i(p, \boldsymbol{u})\frac{\partial P(z)_-}{\partial p} = -\left(\frac{\partial P(z)_+}{\partial p}R_i\right)_- .$$

Hence, if S_- is the solution of (23) associated with F_i then $\tilde{S}_- := S_- - P(z)_-$ is the solution of (23) associated with $\tilde{F}_i$. It is easy to see that the transformation (27) describes translational symmetries of the implicit relations (26)

$$\tilde{\boldsymbol{u}}(\boldsymbol{t}) := \boldsymbol{u}(\boldsymbol{t} + \boldsymbol{c}) \tag{28}$$

where $\boldsymbol{c} := (c_1, c_2, \ldots)$ are the coefficients of the Taylor expansion of P

$$P(z) = \sum_{n\geqslant 0} c_n z^n.$$

In [15–19] inverse problem techniques are used to construct the S-functions for solving the initial value problem of several dispersionless models. Our analysis provides an alternative viewpoint for determining S which is based on the systems of differential equations (16) and (23). Thus, S is characterized by a set of *spectral data*$\{p_i(\boldsymbol{u}), r_{ij}(\boldsymbol{u}), F_i(\boldsymbol{u}) : 1 \leqslant i, j \leqslant N\}$. Moreover, from (18) one finds that the compatibility conditions (19) and (24) are equivalent to the following consistency conditions for the spectral data

$$\begin{aligned}
&r_{ik}\frac{\partial p_k}{\partial u_j} - r_{jk}\frac{\partial p_k}{\partial u_i} = \sum_{l\neq k} \frac{r_{jl}r_{ik} - r_{il}r_{jk}}{p_k - p_l} \\
&\frac{\partial r_{ik}}{\partial u_j} - \frac{\partial r_{jk}}{\partial u_i} = 2\sum_{l\neq k} \frac{r_{jk}r_{il} - r_{ik}r_{jl}}{(p_k - p_l)^2} \\
&r_{ik}\frac{\partial F_k}{\partial u_j} - r_{jk}\frac{\partial F_k}{\partial u_i} = \sum_{l\neq k} \frac{r_{jl}r_{ik} - r_{il}r_{jk}}{(p_k - p_l)^2}(F_k - F_l)
\end{aligned} \tag{29}$$

where $i \neq j$. In this way the first two groups of equations of the system (29) characterize the reductions of the dKP hierarchy, while the whole system determines the set of hodograph solutions.

2.2.1. Differential form formulation of (29): compatibility. The equations (29) can be neatly written in terms of differential forms. For that aim we introduce the following notation

$$\varrho_k := \sum_{i=1}^{N} r_{ik}\, \mathrm{d}u_i$$

so that (29) are equivalent to

$$\begin{aligned}
\mathrm{d}(p_k\varrho_k) &= \varrho_k \wedge \sum_{l\neq k} \frac{p_k + p_l}{(p_k - p_l)^2}\varrho_l \\
\mathrm{d}\varrho_k &= 2\varrho_k \wedge \sum_{l\neq k} \frac{1}{(p_k - p_l)^2}\varrho_l \\
\mathrm{d}(F_k\varrho_k) &= \varrho_k \wedge \sum_{l\neq k} \frac{F_k + F_l}{(p_k - p_l)^2}\varrho_l .
\end{aligned} \tag{30}$$

We shall show that for any solution $\{p_k, F_k, \varrho_k\}$ of (30) the following equations are satisfied

$$\mathrm{d}\left(\varrho_k \wedge \sum_{l\neq k} \frac{1}{(p_k - p_l)^2}\varrho_l\right) = 0 \tag{31}$$

$$\mathrm{d}\left(\varrho_k \wedge \sum_{l\neq k} \frac{p_k + p_l}{(p_k - p_l)^2}\varrho_l\right) = 0 \tag{32}$$

$$\mathrm{d}\left(\varrho_k \wedge \sum_{l\neq k} \frac{F_k + F_l}{(p_k - p_l)^2}\varrho_l\right) = 0. \tag{33}$$

It is enough to check (33), as (31) and (32) follow from it by choosing $F_k = 1/2$ and $F_k = p_k$, respectively. One easily gets the desired result as follows:

$$\mathrm{d}\left(\varrho_k \wedge \sum_{l\neq k} \frac{F_k + F_l}{(p_k - p_l)^2}\varrho_l\right) = -2\varrho_k \wedge \sum_{l,m\neq k} S_{klm}\varrho_l \wedge \varrho_m = 0.$$

The last equality is a consequence of the skew symmetry of the wedge product: $\varrho_l \wedge \varrho_m = -\varrho_m \wedge \varrho_l$ and symmetry of the coefficient

$$\begin{aligned}
S_{klm} = {} & \frac{\left(2p_k^2 + p_m^2 + p_l^2 - 2(p_l + p_m)p_k\right) F_k}{(-p_l + p_m)^2(p_k - p_m)^2(p_k - p_l)^2} + \frac{\left(p_k^2 + 2p_l^2 + p_m^2 - 2p_l p_k - 2p_l p_m\right) F_l}{(-p_l + p_m)^2(p_k - p_m)^2(p_k - p_l)^2} \\
& + \frac{\left(p_k^2 + p_l^2 + 2p_m^2 - 2p_k p_m - 2p_l p_m\right) F_m}{(-p_l + p_m)^2(p_k - p_m)^2(p_k - p_l)^2}
\end{aligned}$$

given by $S_{klm} = S_{kml}$.

The system (31)–(33) means that the system itself ensures the equality of cross-derivatives. Thus, we conclude that the system (29) is compatible in the sense that

$$\begin{aligned}
\frac{\partial}{\partial u_m}\frac{\partial p_k}{\partial u_l} &= \frac{\partial}{\partial u_l}\frac{\partial p_k}{\partial u_m} \\
\frac{\partial}{\partial u_m}\frac{\partial r_{ik}}{\partial u_l} &= \frac{\partial}{\partial u_l}\frac{\partial r_{ik}}{\partial u_m} \\
\frac{\partial}{\partial u_m}\frac{\partial F_k}{\partial u_l} &= \frac{\partial}{\partial u_l}\frac{\partial F_k}{\partial u_m}
\end{aligned}$$

holds in virtue of the equations (29).

2.2.2. Bourlet analysis. Our first aim is to show that (29) has a number of redundant equations. We shall concentrate on the equations

$$r_{ik}\frac{\partial p_k}{\partial u_j} - r_{jk}\frac{\partial p_k}{\partial u_i} = \sum_{l\neq k}\frac{r_{jl}r_{ik} - r_{il}r_{jk}}{p_k - p_l} \tag{34}$$

$$r_{ik}\frac{\partial F_k}{\partial u_j} - r_{jk}\frac{\partial F_k}{\partial u_i} = \sum_{l\neq k}\frac{r_{jl}r_{ik} - r_{il}r_{jk}}{(p_k - p_l)^2}(F_k - F_l). \tag{35}$$

For each k we define $s_k \in \{1, \ldots, N\}$ by the condition $r_{s_k k} \neq 0$ and $r_{ik} = 0$ for $i > s_k$. Then, (34) imply

$$\frac{\partial p_k}{\partial u_i} = \frac{1}{r_{s_k k}}\left(r_{ik}\frac{\partial p_k}{\partial u_{s_k}} - \sum_{l\neq k}\frac{r_{s_k l}r_{ik} - r_{il}r_{s_k k}}{p_k - p_l}\right) \qquad i \neq s_k. \tag{36}$$

Moreover, (34) for $i, j \neq s_k$ holds whenever (36) is satisfied:

$$\begin{aligned}
r_{ik}\frac{\partial p_k}{\partial u_j} - r_{jk}\frac{\partial p_k}{\partial u_i} &= \frac{r_{ik}}{r_{s_k k}}\left(r_{jk}\frac{\partial p_k}{\partial u_{s_k}} - \sum_{l\neq k}\frac{r_{s_k l}r_{jk} - r_{jl}r_{s_k k}}{p_k - p_l}\right)\\
&\quad - \frac{r_{jk}}{r_{s_k k}}\left(r_{ik}\frac{\partial p_k}{\partial u_{s_k}} - \sum_{l\neq k}\frac{r_{s_k l}r_{ik} - r_{il}r_{s_k k}}{p_k - p_l}\right)\\
&= \sum_{l\neq k}\frac{r_{jl}r_{ik} - r_{il}r_{jk}}{p_k - p_l}.
\end{aligned}$$

Second, we notice that when $r_{s_k k} \neq 0$, (35) implies

$$\frac{\partial F_k}{\partial u_i} = \frac{1}{r_{s_k k}}\left(r_{ik}\frac{\partial F_k}{\partial u_{s_k}} - \sum_{l\neq k}\frac{r_{s_k l}r_{ik} - r_{il}r_{s_k k}}{(p_k - p_l)^2}(F_k - F_l)\right) \qquad i \neq s_k. \tag{37}$$

As before (35) for $i, j \neq s_k$ holds whenever (37) is satisfied:

$$\begin{aligned}
r_{ik}\frac{\partial F_k}{\partial u_j} - r_{jk}\frac{\partial F_k}{\partial u_i} &= \frac{r_{ik}}{r_{s_k k}}\left(r_{jk}\frac{\partial F_k}{\partial u_{s_k}} - \sum_{l\neq k}\frac{r_{s_k l}r_{jk} - r_{jl}r_{s_k k}}{(p_k - p_l)^2}(F_k - F_l)\right)\\
&\quad - \frac{r_{jk}}{r_{s_k k}}\left(r_{ik}\frac{\partial F_k}{\partial u_{s_k}} - \sum_{l\neq k}\frac{r_{s_k l}r_{ik} - r_{il}r_{s_k k}}{(p_k - p_l)^2}(F_k - F_l)\right)\\
&= \sum_{l\neq k}\frac{r_{jl}r_{ik} - r_{il}r_{jk}}{(p_k - p_l)^2}(F_k - F_l).
\end{aligned}$$

Then, the system (29) is equivalent to

$$\begin{aligned}
\frac{\partial p_k}{\partial u_i} &= \frac{1}{r_{s_k k}}\left(r_{ik}\frac{\partial p_k}{\partial u_{s_k}} - \sum_{l\neq k}\frac{r_{s_k l}r_{ik} - r_{il}r_{s_k k}}{p_k - p_l}\right) \qquad i < s_k\\
\frac{\partial p_k}{\partial u_i} &= -\frac{1}{r_{s_k k}}\sum_{l\neq k}\frac{r_{s_k l}r_{ik} - r_{il}r_{s_k k}}{p_k - p_l} \qquad i > s_k\\
\frac{\partial F_k}{\partial u_i} &= \frac{1}{r_{s_k k}}\left(r_{ik}\frac{\partial F_k}{\partial u_{s_k}} - \sum_{l\neq k}\frac{r_{s_k l}r_{ik} - r_{il}r_{s_k k}}{(p_k - p_l)^2}(F_k - F_l)\right) \qquad i < s_k
\end{aligned} \tag{38}$$

$$\frac{\partial F_k}{\partial u_i} = -\frac{1}{r_{s_k k}} \sum_{l \neq k} \frac{r_{s_k l} r_{ik} - r_{il} r_{s_k k}}{(p_k - p_l)^2} (F_k - F_l) \qquad i > s_k$$

$$\frac{\partial r_{ik}}{\partial u_j} = \frac{\partial r_{jk}}{\partial u_i} + 2 \sum_{l \neq k} \frac{r_{jk} r_{il} - r_{ik} r_{jl}}{(p_k - p_l)^2} \qquad i > j$$

for $k = 1, \ldots, N$. The system written in this form is of Bourlet type [21]. In this sense $\left(u_1, \ldots, u_{s_k-1}, u_{s_k+1}, u_{N-1}\right)$ are principal variables for p_k, F_k while u_{s_k} are parametric variables. For r_{ik} we have that $(u_1, \ldots, u_{i-1})$ are principal variables and $(u_i, \ldots, u_N)$ are parametric variables. The compatibility in principal variables is ensured from the result of the previous section which gives compatibility among all variables. To apply the Bourlet theorem we should check the analytic character of the functions defining the system. We see that once the conditions $r_{s_k k} \neq 0$ and $p_k \neq p_l$ are ensured the analytic requirement is satisfied. Following Bourlet, we conclude that there is a *unique* solution $\{p_k, F_k, r_{ik}\}$ in a neighbourhood of an initial point $\boldsymbol{u}_0 = \left(u_1^{(0)}, \ldots, u_N^{(0)}\right)$ such that when the principal variables assume initial values then the solution is transformed into a set of arbitrary analytic functions of the corresponding parametric variables. Thus, the general solution will depend on $N(N+1)$ arbitrary analytic functions of the parametric variables, $3N$ of one variable, and for each $l = 2, \ldots, N-1$ there are N analytical functions of l variables.

3. Hodograph solutions and systems of hydrodynamic type

3.1. Associated systems of hydrodynamic type

The implicit equations (26) are transformations of hodograph type which reveal the presence of an underlying system of HT equations. In fact from (17), provided $z = z(p, \boldsymbol{u})$ is regular at the points p_i, it follows that

$$\frac{\partial z}{\partial p}(p_i) = 0$$

so that (1) implies

$$\sum_{j=1}^{N} \frac{\partial z_i}{\partial u_j} \frac{\partial u_j}{\partial t_n} = v_{in} \sum_{j=1}^{N} \frac{\partial z_i}{\partial u_j} \frac{\partial u_j}{\partial x} \qquad n \geqslant 1$$

where

$$z_i := z(p_i, \boldsymbol{u}(\boldsymbol{t})).$$

Thus, by expressing $\boldsymbol{u}(\boldsymbol{t})$ in terms of the functions z_i, we find that the functions $\boldsymbol{u}(\boldsymbol{t})$ satisfy the system of equations of hydrodynamic type

$$\frac{\partial \boldsymbol{u}}{\partial t_n} = A_n(\boldsymbol{u}) \frac{\partial \boldsymbol{u}}{\partial x} \qquad n = 1, \ldots, N$$

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \qquad A_n := K^{-1} D_n K \tag{39}$$

$$D_n := \mathrm{diag}(v_{1n}, \ldots, v_{Nn}) \qquad K_{ij} := \frac{\partial z_i}{\partial u_j}.$$

Note that, by taking into account that $v_{2i} = 2p_i$, from (39) we obtain the Gibbons–Kodama formula [6]

$$A_n = v_n(A) \qquad A := A_2/2 \tag{40}$$

where $v_n(p) := \frac{\partial\Omega_n}{\partial p}$. This result shows the particular relevance of the $n = 2$ flow (Benney moment equations) in the analysis of reductions of the dKP hierarchy. Furthermore, by using the HT equation $\boldsymbol{u}_t = A(\boldsymbol{u})\boldsymbol{u}_x$ associated with the Benney moment equations we may rewrite (1) for $n = 2$ as

$$\sum_j \left(\sum_i A_{ij}\frac{\partial z}{\partial u_i} - p\frac{\partial z}{\partial u_j} + \frac{\partial z}{\partial p}\frac{\partial a_1}{\partial u_j}\right)\frac{\partial u_j}{\partial x} = 0.$$

Hence, if we assume that the functions $\partial_x u_j,\ j = 1, \ldots, N$ are independent, we conclude that the functions R_i in (16) and (17) can be expressed as

$$R_i(p, \boldsymbol{u}) = \sum_{j=1}^{N} (A(\boldsymbol{u}) - p)^{-1}_{ji}\frac{\partial a_1}{\partial u_j} \tag{41}$$

Therefore, the compatibility condition (19) for the reductions of the dKP hierarchy can be formulated in terms of the matrix A associated with the Benney system.

From (41) we deduce that

$$r_{ik} = -\frac{\partial z_k}{\partial u_i}r_k \qquad r_k := \frac{\partial a_1}{\partial z_k}.$$

Thus, the differential forms ϱ_k are

$$\varrho_k = -\sum_{i=1}^{N}\frac{\partial z_k}{\partial u_i}\frac{\partial a_1}{\partial z_k}\,\mathrm{d}u_i = -r_k\,\mathrm{d}z_k.$$

In terms of the new coordinates $\{z_i\}_{i=1}^N$ the system (30) reads

$$\frac{\partial r_i}{\partial z_j} = 2\frac{r_i r_j}{(p_j - p_i)^2}$$
$$\frac{\partial p_i}{\partial z_j} = \frac{r_j}{p_j - p_i}$$
$$\frac{\partial F_i}{\partial z_j} = r_j\frac{F_j - F_i}{(p_j - p_i)^2}.$$

We note that according to (50)

$$\frac{\partial F_i}{\partial z_j}\frac{1}{F_j - F_i} = \frac{\partial p_i}{\partial z_j}\frac{1}{p_j - p_i} = \frac{1}{2}\frac{\partial \ln r_i}{\partial z_j} \qquad i \neq j. \tag{42}$$

These relations provide a link between the system (29) and the theory of Comberscure transformations of symmetric conjugate nets [22]. Thus, if we define

$$\beta_{ij} := \frac{1}{\sqrt{r_i}}\frac{\partial\sqrt{r_j}}{\partial z_i} = \frac{\sqrt{r_i r_j}}{(p_i - p_j)^2} = \beta_{ji} \qquad i \neq j \tag{43}$$

then there exists a family of parallel conjugate nets $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{u})$ given by the solutions of

$$\frac{\partial \boldsymbol{x}}{\partial z_i} = H_i\boldsymbol{X}_i \tag{44}$$

where H_i and $\boldsymbol{X}_i$ (the Lamé and renormalized tangent vectors, respectively) are characterized by the equations

$$\frac{\partial H_i}{\partial z_i} = \beta_{ji}H_j \tag{45}$$

and

$$\frac{\partial \boldsymbol{X}_i}{\partial z_i} = \beta_{ij}\boldsymbol{X}_j. \tag{46}$$

Obviously, $H_i := \sqrt{r_i}$ solves (45) and as a consequence one proves that (42) means that $F_i H_i$ and $p_i H_i$ are also solutions of (45).

3.2. Diagonal reductions

From (1) it follows that

$$\frac{\partial z_i}{\partial t_n} = v_{in} \frac{\partial z_i}{\partial x} \tag{47}$$

so that $z_i = z_i(\boldsymbol{u})$ constitute a set of Riemann invariants of the HT system (39). If we take $\boldsymbol{z} = (z_1(\boldsymbol{u}), \ldots, z_N(\boldsymbol{u}))$ as the new dependent variables of the N-reduction the associated HT system is (47), so that the A-matrix for the Benney flow is $A_{ij} = p_i \delta_{ij}$. Hence, by using (41) we get that $p = p(z, \boldsymbol{u}(\boldsymbol{z}))$ satisfy

$$\frac{\partial p}{\partial z_i} = -\frac{r_i}{p - p_i(\boldsymbol{z})} \qquad r_i := \frac{\partial a_1}{\partial z_i}. \tag{48}$$

These equations were already found by Gibbons–Tsarev [8, 9] in their analysis of the consistency conditions of reductions of the Benney moment equations in characteristic form.

Reciprocally, if we consider the reductions determined by systems of the form

$$\frac{\partial p}{\partial u_i} = -\frac{r_i}{p - p_i(\boldsymbol{u})} \qquad r_i := \frac{\partial a_1}{\partial u_i} \tag{49}$$

then

$$\frac{\partial z}{\partial u_i} = \frac{r_i(\boldsymbol{u})}{p - p_i} \frac{\partial z}{\partial p}$$

so that $z_j(\boldsymbol{u}) = z(p_j, \boldsymbol{u})$ satisfies

$$\frac{\partial z_j}{\partial u_i} = 0 \qquad i \neq j$$

and therefore each u_j is a function of z_j only. This means that the systems of the form (49) determine those reductions of the dKP hierarchy in which $\boldsymbol{u}$ evolve according to diagonal HT systems. Henceforth these reductions will be referred to as *diagonal reductions*. Since every reduction is associated with an HT system which adopts a diagonal form under the change of variables $\boldsymbol{u} \to \boldsymbol{z}$, classifying diagonal reductions would allow us to classify the whole class of reductions of the dKP hierarchy.

The compatibility conditions (29) for diagonal reductions and their corresponding hodograph reductions read

$$\begin{aligned}
\frac{\partial r_i}{\partial u_j} &= 2\frac{r_i r_j}{(p_j - p_i)^2} \\
\frac{\partial p_i}{\partial u_j} &= \frac{r_j}{p_j - p_i} \\
\frac{\partial F_i}{\partial u_j} &= r_j \frac{F_j - F_i}{(p_j - p_i)^2}
\end{aligned} \tag{50}$$

where $i \neq j$.

This is a compatible system of first-order differential equations with a solution depending on $3N$ arbitrary functions of one variable. The first two groups of equations were found by Gibbons and Tsarev [7, 8] in their analysis of the reductions of the Benney equations. We also remark that the geometrical interpretation described above obviously holds here as well.

4. Examples

4.1. N = 1 reductions

If only one function $u = u(\boldsymbol{t})$ is assumed to be involved in the reduction and it is set $u = -a_1$, then (16) becomes Abel's equation

$$\frac{\partial p}{\partial u} = \frac{1}{p - p_1(u)} \tag{51}$$

and from (17) we get the following recursion relation for the coefficients of the expansion of $z = z(p, u)$

$$a_1 = -u \qquad a_2 = -\int p_1(u)\,\mathrm{d}u$$
$$a'_{m+2} = p_1(u)a'_{m+1} + ma_m \qquad m \geqslant 1$$

where $a'_m := \frac{\partial a_m}{\partial u}$. We can now use this expansion to generate solutions of the equations of the dKP hierarchy. For instance, by setting $t_n = 0,\ n > 4$, in (26) we get the following implicit equation for determining u

$$4\left(p_1(u)^3 - 2up_1(u) - \int p_1(u)\,u\right)t_4 + 3(p_1(u)^2 - u)t_3 + 2p_1(u)t_2 + x = -F(u) \tag{52}$$

where $p_1(u)$ and $F(u)$ are arbitrary functions. For $t_4 = 0$ this result reduces to Kodama's equation [10, 11] for $N = 1$ reductions of the dispersionless KP equation (5).

An explicit expression for the solution $z = z(p, u)$ of (17) is available in a few cases only. For instance

1. $p_1(u) \equiv 0$ (dKdV-reduction)
$$z = (p^2 - 2u)^{\frac{1}{2}}.$$
2. $p_1(u) = u^{\frac{1}{2}}$
$$z = \left(p^3 - 3up - 2u^{\frac{3}{2}}\right)^{\frac{1}{3}}.$$
3. $p_1(u) = u$
$$z = 1 + W\left(\mathrm{e}^{p-1}(p - u - 1)\right)$$
where $W = W(y)$ (Lambert function) is the inverse function of $y = x\,\mathrm{e}^x$.
4. $p_1(u) = u^2$
$$z = \frac{3}{4\mathrm{i}}\left(\ln\frac{\mathrm{Ai}^{(-)}(p) - u\,\partial_p\mathrm{Ai}^{(-)}(p)}{\mathrm{Ai}^{(+)}(p) - u\,\partial_p\mathrm{Ai}^{(+)}(p)} - \mathrm{i}\frac{\pi}{2}\right)$$
where $\mathrm{Ai}^{(\pm)}$ are the Airy functions
$$\mathrm{Ai}^{(\pm)}(p) := \mathrm{Bi}(-p) \pm \mathrm{i}\,\mathrm{Ai}(-p).$$

In what concerns the determination of S_- for $p_1 = 0$ we have that (23) is now

$$\frac{\partial S_-(p)}{\partial u} + \frac{1}{p}\frac{\partial S_-(p)}{\partial p} = \frac{F}{p}.$$

An explicit solution is given by

$$S_-(p, u) = -\left(\int_0^{z(p,u)} F\left(\frac{1}{2}(q^2 - z(p, u)^2)\right)\mathrm{d}q\right)_-.$$

4.2. $N = 2$ reductions

Let us consider now the case $\boldsymbol{u} = (u, v)$ with $u = -a_1$. From (41) we get

$$\begin{aligned}\frac{\partial p}{\partial u} &= \frac{p - A_{22}}{(p - A_{11})(p - A_{22}) - A_{12}A_{21}}\\ \frac{\partial p}{\partial v} &= \frac{A_{12}}{(p - A_{11})(p - A_{22}) - A_{12}A_{21}}\end{aligned} \tag{53}$$

where $A := (A_{ij}(\boldsymbol{u}))$ is the 2×2 matrix function associated with the Benney flow. The right-hand sides of (53) have simple poles at

$$A_\pm := \tfrac{1}{2}\left(\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\ \det A}\ \right).$$

In this case (19) leads to the following conditions:

$$\partial_v A_{11} = \partial_u A_{12} \qquad \begin{pmatrix}\partial_v \det A\\ -\partial_u(u + \det A)\end{pmatrix} = A\begin{pmatrix}\partial_v\ \operatorname{tr} A\\ -\partial_u\ \operatorname{tr} A\end{pmatrix}. \tag{54}$$

The moments of $z(p, \boldsymbol{u})$ are determined by the recursion relations

$$\begin{aligned}&a_1 = -u \qquad a_2 = -\int A_{11}\,\mathrm{d}u + A_{12}\,\mathrm{d}v\\ &a_3 = \int (\det A - u - A_{11}\ \operatorname{tr} A)\,\mathrm{d}u - A_{12}\ \operatorname{tr} A\,\mathrm{d}v\\ &\partial_u a_{m+2} = \operatorname{tr} A\ \partial_u a_{m+1} - \det A\ \partial_u a_m + m a_m - (m-1)A_{22}a_{m-1}\\ &\partial_v a_{m+2} = \operatorname{tr} A\ \partial_v a_{m+1} - \det A\ \partial_v a_m + (m-1)A_{12}a_{m-1}.\end{aligned}$$

If we denote

$$F_\pm(\boldsymbol{u}) := \frac{\partial S_-(p)}{\partial p}\ |_{A_\pm}$$

then (24) reduces to

$$(p - A_{22})\partial_v\ E - A_{12}\partial_u\ E = (p^2 - p\ \operatorname{tr} A - \det A)\partial_v\left(\frac{F_+ - F_-}{A_+ - A_-}\right)$$

where E is taken as

$$E := p\frac{F_+ - F_-}{A_+ - A_-} + \frac{A_+F_- - A_-F_+}{A_+ - A_-}.$$

Thus, one finds at once that

$$\begin{pmatrix}-\partial_v\ F\\ \partial_u\ F\end{pmatrix} = A\begin{pmatrix}\partial_v\ G\\ -\partial_u\ G\end{pmatrix}. \tag{55}$$

where

$$F := \frac{A_-F_+ - A_+F_-}{A_+ - A_-} \qquad G := \frac{F_- - F_+}{A_+ - A_-}.$$

Hence if A and $F_\pm$ verify their corresponding consistency conditions and we set $t_n = 0,\ n > 4$, then a solution of the first flows of the dKP hierarchy can be found by solving the following implicit equations for $\boldsymbol{u}$:

$$4\left(A_\pm^3 - 2uA_\pm - \int A_{11}\,u + A_{12}\,v\right)t_4 + 3\left(A_\pm^2 - u\right)t_3 + 2A_\pm t_2 + x = -F_\pm. \tag{56}$$

If $t_4 = 0$ these equations are equivalent to the Kodama system for $N = 2$ reductions [10, 11]

$$\begin{aligned}&-3(u + \det A)t_3 + x = F\\ &3\operatorname{tr} A\ t_3 + 2t_2 = G.\end{aligned} \tag{57}$$

A particularly interesting case arises by imposing $u = -a_1,\ v = -a_2$ which corresponds to the choice

$$A = \begin{pmatrix} 0 & 1 \\ -V & W \end{pmatrix} \qquad V := A_+A_- \qquad W := A_+ + A_-.$$

Thus one finds that (54) becomes

$$\begin{aligned} &\partial_v V + \partial_u W = 0 \\ &\partial_u V - V\ \partial_v W + W\ \partial_v V + 1 = 0. \end{aligned} \tag{58}$$

Hence by setting

$$V = \partial_u Z \qquad W = -\partial_v Z$$

(58) can be formulated as a Monge–Ampere equation

$$\partial_{uu} Z + \partial_u Z\ \partial_{vv} Z - \partial_v Z\ \partial_{uv} Z + 1 = 0. \tag{59}$$

Analogously, (55) can be written as

$$\begin{aligned} &F = \partial_u T \qquad G = \partial_v T \\ &\partial_{uu} T + V\ \partial_{vv} T + W\ \partial_{uv} T = 0. \end{aligned} \tag{60}$$

Next, we will construct some solutions of the dKP equation.

A solution of (58) and (60) is given by

$$W = \frac{2v}{u} \qquad V = \frac{v^2}{u^2} + cu^2 + u \qquad T = k_1 u + k_2 v$$

The corresponding hodograph solutions for (5) are given by

$$\begin{aligned} u(x, y, t) &= \frac{1}{6ct}\left(-6t + \sqrt{36t^2 + c\big[12t(x - k_1) - (2y - k_2)^2\big]}\right) \\ u(x, y, t) &= \frac{12t(x - k_1) - (2y - k_2)^2}{72t^2} \end{aligned}$$

which correspond to $c \neq 0$ and $c = 0$, respectively.

Another interesting solution of (58) and (60) is

$$W = \frac{2v}{u} \qquad V = \frac{v^2}{u^2} + u \qquad T = k\frac{v}{u}.$$

It leads to a hodograph solution of (5) implicitly defined by the algebraic equation

$$72t^2u^3 + 4(y^2 - 3tx)u^2 = k^2.$$

4.3. N = 3 reductions

Let us now denote $\boldsymbol{u} = (u, v, w)$ and consider the system

$$\begin{aligned} \frac{\partial p}{\partial u} &= \frac{p^2 + B_1 p + B_2}{(p - A_1)(p - A_2)(p - A_3)} \\ \frac{\partial p}{\partial v} &= \frac{p + C_1}{(p - A_1)(p - A_2)(p - A_3)} \\ \frac{\partial p}{\partial w} &= \frac{D_1}{(p - A_1)(p - A_2)(p - A_3)}. \end{aligned} \tag{61}$$

As the computations in this case are very involved it is convenient to assume that (u, v, w) are given by the first coefficients of the expansion of $p = p(z, \boldsymbol{u})$

$$p = z + \frac{u}{z} + \frac{v}{z^2} + \frac{w}{z^3} + O\left(\frac{1}{z^4}\right)$$

thus we have

$$B_1 = C_1 = -V \qquad B_2 = R + u \qquad D_1 = 1$$

where

$$\begin{aligned} V &= A_1 + A_2 + A_3 \\ R &= A_1A_2 + A_2A_3 + A_3A_1 \\ H &= A_1A_2A_3. \end{aligned}$$

The compatibility conditions (19) can be formulated as

$$\begin{aligned} V_v &= -R_w \\ R_v &= -H_w + RV_w - VR_w \\ H_v &= 1 - VH_w + HV_w \\ V_u &= H_w + uV_w \\ R_u &= VH_w - HV_w + uR_w - 2 \\ H_u &= -V + RH_w - HR_w + uH_w. \end{aligned} \tag{62}$$

Now, if we take $S = S_+(p)$ and $t_n = 0$, $n > 4$, then (21) implies

$$\begin{aligned} \frac{\partial S_+(p)}{\partial p} &= 4t_4(p - A_1)(p - A_2)(p - A_3) \\ &= 4t_4(p^3 - Vp^2 + Rp - H). \end{aligned} \tag{63}$$

On the other hand, as $a_1 = -b_1 = -u$, $a_2 = -b_2 = -v$, we have

$$\frac{\partial S_+(p)}{\partial p} = x + 2py + 3(p^2 - u)t + 4(p^3 - 2up - v)t_4. \tag{64}$$

Thus, by comparing (63) and (64) we find that the solutions for the first two members of the dKP hierarchy can be obtained by solving the system

$$V = -\frac{3t}{4t_4} \qquad R = \frac{y}{2t_4} - 2u \qquad H = v - \frac{x - 3tu}{4t_4}. \tag{65}$$

For instance, by trying a function V of the form $V = V(u, v)$ we find a solution of (62) given by

$$\begin{aligned} V &= k_1v + k_2u + k_3 \\ R &= k_4 + (k_2k_3 - 2)u + \tfrac{1}{2}\left(k_2^2 - k_1\right)u^2 + (k_1k_3 - k_2)v + \tfrac{1}{2}k_1^3v^2 - k_1w + k_1k_2uv \\ H &= k_5e^{k_1u} + \left(\frac{k_2}{2} - \frac{k_2^3}{2k_1}\right)u^2 + \frac{3k_1k_2 - k_2^3 - k_1k_2^2k_3}{k_1^2}u - \frac{1}{2}k_1k_2v^2 \\ &\quad + (1 - k_2k_3)v + k_2w - k_2^2uv - \frac{k_2^3}{k_1^3} - \frac{k_2k_4}{k_1} + \frac{k_3}{k_1} + \frac{3k_2}{k_1^2} - \frac{k_2^2k_3}{k_1^2} \end{aligned}$$

where $k_1,\ k_2,\ k_3,\ k_4,\ k_5$ are arbitrary constants with $k_1 \neq 0$. Hence we have a solution of (5) and (6) implicitly defined by the transcendent equation

$$k_1^3x - 2k_1^2k_2y + 3k_1k_2^2t + 4\left(k_1^2k_3 + 3k_1k_2 - k_2^3\right)t_4 + \left(12k_1^2k_2t_4 - 3k_1^3t\right)u + 4k_1^3k_5t_4e^{k_1u} = 0$$

and

$$v = -\frac{k_3}{k_1} - \frac{3t}{4k_1 t_4} - \frac{k_2}{k_1}u.$$

In the particular case $k_5 = 0$ one finds

$$u = \frac{k_1^3 x - 2k_1^2 k_2 y + 3k_1 k_2^2 t + 4\left(k_1^2 k_3 + 3k_1 k_2 - k_2^3\right) t_4}{3k_1^2 (k_1 t - 4k_2 t_4)}.$$

4.4. Diagonal reductions

We have seen above that the diagonal reductions

$$\frac{\partial p}{\partial u_i} = -\frac{r_i}{p - p_i(\boldsymbol{u})} \qquad r_i := \frac{\partial a_1}{\partial u_i}$$

and their corresponding hodograph solutions are described by the compatible system of first-order differential equations (50). In [8] Gibbons and Tsarev provide a set of solutions for the first two subsystems of (50) which are both scaling and Galilean invariants. They are defined by

$$2\sum_{j\neq i} \frac{u_j - u_i}{(p_j - p_i)^2} r_j = 1 \qquad p_i = u_i + \sum_{j\neq i} \frac{u_j - u_i}{p_j - p_i} r_j. \tag{66}$$

Corresponding solutions of the third subsystem of (50) satisfying the invariance properties

$$\sum_i u_i \frac{\partial F_j}{\partial u_i} = F_j \qquad \sum_i \frac{\partial F_j}{\partial u_i} = 1$$

are determined by

$$F_i = u_i + \frac{1}{2}\sum_{j\neq i} (u_j - u_i)(F_j - F_i)\frac{\partial \ln r_i}{\partial u_j}. \tag{67}$$

Let us analyse the case $N = 2$ in closer detail. From (66) we may start with a scaling and Galilean invariant choice for r_j and p_j

$$r_1 = -r_2 = \tfrac{1}{8}(u_1 - u_2)$$
$$p_1 = \tfrac{1}{4}(3u_1 + u_2) \qquad p_2 = \tfrac{1}{4}(u_1 + 3u_2).$$

The conditions for F_j become

$$\frac{\partial F_1}{\partial u_2} = \frac{\partial F_2}{\partial u_1} = \frac{1}{2}\frac{F_1 - F_2}{u_1 - u_2}$$

which are equivalent to

$$F_i = \frac{\partial U}{\partial u_i} \qquad i = 1, 2$$
$$2(u_1 - u_2)\frac{\partial^2 U}{\partial u_1 \partial u_2} = \frac{\partial U}{\partial u_1} - \frac{\partial U}{\partial u_2}.$$

The solution of the equation for U can be found by the method of separation of variables and is a superposition of functions of the form

$$(a\, J_0(k(u_1 - u_2)) + b\, Y_0(k(u_1 - u_2)))\, (c\, \cos(k(u_1 + u_2)) + d\, \sin(k(u_1 + u_2)))$$

where J_0 and Y_0 are the standard Bessel functions. We find also the simple solution

$$U = c\, \ln(u_1 - u_2) \qquad F_1 = -F_2 = \frac{c}{u_1 - u_2}$$

which leads to the hodograph relations

$$3\left(\tfrac{1}{16}(3u_1+u_2)^2+a_1\right)t_3+\tfrac{1}{2}(3u_1+u_2)t_2+x=\frac{c}{u_2-u_1}$$
$$3\left(\tfrac{1}{16}(u_1+3u_2)^2+a_1\right)t_3+\tfrac{1}{2}(u_1+3u_2)t_2+x=\frac{c}{u_1-u_2}$$

where

$$a_1=\tfrac{1}{16}(u_1-u_2)^2.$$

Acknowledgments

This work was completed during the stay of the authors at the Isaac Newton Institute for the Mathematical Sciences, Cambridge University. We would like to thank the organisers of the program Integrable Systems for the kind invitation. L Martínez Alonso is grateful to the Fundation Banco Bilbao Vizcaya Argentaria for its support. The authors also acknowledge G Álvarez Galindo for his help in computing the Airy potential function.

References

[1] Lax P D and Levermore C D 1983 *Commun. Pure. Appl. Math.* **36** 253, 571, 809
[2] Tian Fei Ran 1993 *Commun. Pure. Appl. Math.* **46** 1093
[3] Krichever I M 1992 *Commun. Pure. Appl. Math.* **47** 437
[4] Krichever I M 1992 *Commun. Math. Phys.* **143** 415
[5] Aoyama S and Kodama Y 1996 *Commun. Math. Phys.* **182** 1185
[6] Kodama Y and Gibbons J 1989 *Phys. Lett.* A **135** 167
[7] Mineev-Weinstein M, Wiegmann P B and Zabrodin A 2000 *Phys. Rev. Lett.* **84** 5106
[8] Gibbons J and Tsarev S P 1996 *Phys. Lett.* A **211** 19
[9] Gibbons J and Tsarev S P 1999 *Phys. Lett.* A **258** 263
[10] Kodama Y 1988 *Prog. Theor. Phys. Suppl.* **95** 184
[11] Kodama Y 1988 *Phys. Lett.* A **129** 223
[12] Kupershmidt B A 1990 *J. Phys. A: Math. Gen.* **23** 871
[13] Takasaky T and Takebe T 1992 *Int. J. Mod. Phys.* A **7** 889
Takasaky T and Takebe T 1995 *Rev. Math. Phys.* **7** 743
[14] Benney D J 1973 *Stud. Appl. Math.* **52** 45
[15] Geogjaev V V 1985 *Sov. Phys.–Dokl.* **30** 840
[16] Geogjaev V V 1994 The quasiclassical limit of the inverse scattering problem method *Singular Limits of Dispersive Waves (NATO ASI series B 320)* ed N Ercolani *et al* (New York: Plenum) p 53
[17] Kodama Y and Gibbons J 1994 Solving dispersionless Lax equations *Singular Limits of Dispersive Waves (NATO ASI Series B 320)* ed N Ercolani *et al* (New York: Plenum) p 61
[18] Kodama Y 1990 *Phys. Lett.* A **147** 477
[19] Yu L 2000 *J. Phys. A: Math. Gen.* **33** 8127
[20] Tsarev S P 1994 On the integrability of the averaged KdV and Benney equations *Singular Limits of Dispersive Waves (NATO ASI Series B 320)* ed N Ercolani *et al* (New York: Plenum) p 112
[21] Bianchi L 1992 *Lezioni di Geometria Differenziale* 3rd edn (Bologna: Zanichelli)
[22] Darboux G 1896 *Leçons sur la Théorie Générale des Surfaces IV* (Paris: Gauthier-Villars) (reprinted 1972 New York: Chelsea Publishing Company)